AMSI 2013: MEASURE THEORY

Extra Solutions A

1 to 19

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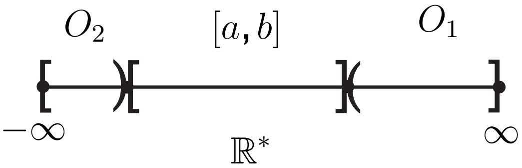
1 We want to show $\mathbb{R}^*$ is compact. So, let $\{O_\alpha\}_\alpha \supseteq \mathbb{R}^*$ be a covering of $\mathbb{R}^*$ by open sets, and we want to show a finite subcollection of these sets also covers $\mathbb{R}^*$. Then one of these sets, call it O_1, contains ∞: and since O_1 is open (i.e. is a union of open intervals), O_1 contains some whole interval $(b, \infty]$ for some $b \in \mathbb{R}$. Similarly, there is an O_2 containing some whole interval $[-\infty, a)$ for some $a \in \mathbb{R}$.



It remains to show that a finite number of the sets $\{O_\alpha\}_\alpha$ covers $[a, b]$. But the sets $\{O_\alpha \sim \{-\infty, \infty\}\}_\alpha$ are open in $\mathbb{R}$ and cover $[a, b]$. (The point is, if $[-\infty, c)$ is open in $\mathbb{R}^*$ then $(-\infty, c)$ is open in $\mathbb{R}$, and similarly for $(c, \infty]$). Since $[a, b]$ is compact in $\mathbb{R}$, a finite subcollection of $\{O_\alpha \sim \{-\infty, \infty\}\}_\alpha$ covers $[a, b]$, and thus a finite subcollection of $\{O_\alpha\}_\alpha$ also covers $[a, b]$.

3 Given measures μ and ν on set X, and $a, b \in [0, \infty]$, we want to show $a\mu + b\nu$ is also a measure on X. It's all very easy, and we'll just prove subadditivity. If $\{A_j\}$ be a countable collection of subsets of X then

$$\begin{aligned}
(a\mu + b\nu)\left(\bigcup_{j=1}^{\infty} A_j\right) &= a\mu\left(\bigcup_{j=1}^{\infty} A_j\right) + b\nu\left(\bigcup_{j=1}^{\infty} A_j\right) \\
&\leqslant a\sum_{j=1}^{\infty} \mu\left(A_j\right) + b\sum_{j=1}^{\infty} \nu\left(A_j\right) \\
&= \sum_{j=1}^{\infty} (a\mu + b\nu)\left(A_j\right) \ .
\end{aligned}$$

4 Given a collection $\{\mu_\alpha\}_{\alpha \in I}$ of measures on a set X, we want to show μ defined by

$$\mu(A) = \sup_{\alpha \in I} \mu_\alpha(A)\,,$$

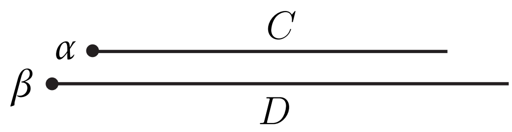
is also a measure on X. Again, we just prove subadditivity. Given a countable collection $\{A_j\}$ of subsets of X we have, for any μ_α,

$$\mu_\alpha\left(\bigcup_{j=1}^{\infty} A_j\right) \leqslant \sum_{j=1}^{\infty} \mu_\alpha\left(A_j\right) \leqslant \sum_{j=1}^{\infty} \mu\left(A_j\right) \ .$$

Taking the *sup* of the LHS over all $\alpha \in I$, we find $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leqslant \sum_{j=1}^{\infty} \mu\left(A_j\right)$, as desired.

5

(a) We want to show that if $C \subseteq D \subseteq \mathbb{R}^*$ then $\inf(C) \geqslant \inf(D)$ and $\sup(C) \leqslant \sup(D)$. Well just show the first inequality.



Let $\alpha = \inf(C)$ and $\beta = \inf(D)$. Since $C \subseteq D$, any lower bound for D is also a lower bound for C; thus β (the greatest lower bound for D) is a lower bound for C. But since α is the *greatest* lower bound for C, we must have $\alpha \geqslant \beta$.

(b) Given $a, b \in \mathbb{R}^*$, and $a \leqslant b + \epsilon$ for every $\epsilon > 0$, we want to show $a \leqslant b$. Suppose not, and thus that $a > b$. This is clearly impossible if $a, b = \pm\infty$. Otherwise, set $\epsilon = \frac{a-b}{2}$, implying

$$a \leqslant b + \epsilon = b + \frac{a-b}{2} = \frac{a}{2} + \frac{b}{2}$$

$$\implies \quad a \leqslant b\,.$$

Thus the hypothesis that $a > b$ in any case leads to the conclusion that $a \leqslant b$.

10 We want to repeat the Cantor set construction, but to wind up with a set D with $\mathscr{L}(D) > 0$. For each n, we choose a t_n with $0 < t_n < 1$. Then, at the n'th stage, $(1 - t_n)$ is the fraction we remove from each interval I_{nj} to form the intervals $I_{n+1,2j-1}$ and $I_{n+1,2j}$. (So, the Cantor set is obtained by setting each $t_n = \frac{2}{3}$). Then, D_n is the union of 2^n intervals I_{nj} with

$$l(I_{nj}) = \frac{t_1}{2} \cdot \frac{t_2}{2} \cdots \frac{t_n}{2} = \frac{1}{2^n} \prod_{i=1}^{n} t_i \qquad j = 1, \ldots, 2^n.$$

Now, we choose the t_n (see below) such that

$$(*) \qquad \prod_{n=1}^{\infty} t_n = \alpha > 0\,,$$

and set $D = \bigcap_{n=1}^{\infty} D_n$. We then want to show $\mathscr{L}(D) = \alpha$.

The $\leqslant$ is immediate from the definition of $\mathscr{L}$, and the equality is immediate if we use Proposition 5(a), Theorem 6 and Theorem 8(b). However, we can also obtain equality using only Proposition 5(a). Notice that all of the open intervals we have removed form a covering of $[0,1]\sim D$. Thus,

$$\mathscr{L}([0,1]\sim D) \leqslant (1-t_1) + t_1(1-t_2) + t_1 \cdot t_2(1-t_3) + \dots$$

But this sum telescopes, and we see

$$\mathscr{L}([0,1]\sim D) \leqslant 1 - \prod_{n=1}^{\infty} t_n = 1 - \alpha\,.$$

Then, by Proposition 5(a) and subadditivity

$$\mathscr{L}(D) \geqslant \mathscr{L}([0,1]) - \mathscr{L}([0,1]\sim D) \geqslant 1 - (1-\alpha) = \prod_{n=1}^{\infty} t_n\,,$$

as desired.

It remains to choose the t_n to satisfy $(*)$, but this is quite easy. Fix $\beta > 0$ and set $t_n = e^{-\frac{\beta}{2^n}}$. Then

$$\prod_{j=1}^{n} t_j = e^{-\beta\left(\sum_{j=1}^{n}\frac{1}{2^j}\right)} \quad\Longrightarrow\quad \prod_{j=1}^{\infty} t_j = e^{-\beta}\,.$$

So, we can arranged for D to have any desired measure $\mathscr{L}(D) = \alpha \in (0,1)$ by setting $\beta = -\log\alpha$.

11 We want to show null sets of a measure μ on a set X are measurable. So, fix $A \subseteq X$ with $\mu(A) = 0$, and consider $B \subseteq X$. Then $B \cap A \subseteq A$ and $B \sim A \subseteq B$. Thus, by monotonicity,

$$\mu(B\cap A) + \mu(B\sim A) \leqslant \mu(A) + \mu(B) = 0 + \mu(B) = \mu(B)\,.$$

12 μ is a measure on X and $B \subseteq X$.

(a) We want to show $\mu \llcorner B$ is a measure. Obviously $\mu \llcorner B(\emptyset) = \mu(B \cap \emptyset) = 0$. For monotonicity, we note that if $A \subseteq C$ then

$$\mu \llcorner B(A) = \mu(B \cap A) \leqslant \mu(B \cap C) = \mu \llcorner B(C) \qquad \text{(since } B \cap A \subseteq B \cap C\text{)}.$$

For countable subadditivity,

$$\mu \llcorner B\left(\bigcup_{j=1}^{\infty} A_j\right) = \mu\left(B \cap \bigcup_{j=1}^{\infty} A_j\right) = \mu\left(\bigcup_{j=1}^{\infty} (B \cap A_j)\right) \leqslant \sum_{j=1}^{\infty} \mu\left(B \cap A_j\right) = \sum_{j=1}^{\infty} \mu \llcorner B\left(A_j\right).$$

(b) Given $A \subseteq X$ is μ-measurable, we want to show that A is $\mu \llcorner B$-measurable. If $C \subseteq X$ then

$$\begin{aligned} \mu \llcorner B(C \cap A) + \mu \llcorner B(C \sim A) &= \mu(B \cap C \cap A) + \mu(B \cap C \sim A) \\ &= \mu(B \cap C) && (A \text{ is } \mu\text{-measurable}) \\ &= \mu \llcorner B(C). \end{aligned}$$

(c) Given B is μ-measurable and $A \subseteq B$ is $\mu \llcorner B$-measurable, we now show A is μ-measurable. If $C \subseteq X$ then

$$\begin{aligned} &\mu(C \cap A) + \mu(C \sim A) \\ &= \mu(C \cap A) + \mu(C \cap B \sim A) + \mu(C \sim A \sim B) && (B \text{ is } \mu\text{-measurable}) \\ &= \mu(B \cap C \cap A) + \mu(C \cap B \sim A) + \mu(C \sim B) && (\text{since } A \subseteq B) \\ &= \mu \llcorner B(C \cap A) + \mu \llcorner B(C \sim A) + \mu(C \sim B) && (\text{definition}) \\ &= \mu \llcorner B(C) + \mu(C \sim B) && (A \text{ is } \mu \llcorner B\text{-measurable}) \\ &= \mu(C) && (B \text{ is } \mu\text{-measurable}) \end{aligned}$$

13 Given a measure μ on a set X, we want to show $\mathcal{N}\mu$ is a σ-algebra, where

$$\mathcal{N}\mu = \{A \subseteq X : \mu(A) = 0 \text{ or } \mu(\sim A) = 0\}\,.$$

Trivially $\emptyset \in \mathcal{N}\mu$, and $\mathcal{N}\mu$ is closed under complements. So, we just have to show $\mathcal{N}\mu$ is closed under countable unions. Let $\{A_j\}$ be a sequence of sets in $\mathcal{N}\mu$.

First suppose that $\mu(A_j) = 0$ for every j. Then, by countable subadditivity,

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leqslant \sum_{j=1}^{\infty} \mu(A_j) = 0.$$

Thus $\bigcup_j A_j \in \mathcal{N}\mu$.

On the other hand, suppose that $\mu(\sim A_n) = 0$ for some n. Then, by de Morgan,

$$\begin{aligned} &\sim\left(\bigcup_{j=1}^{\infty} A_j\right) = \bigcap_{j=1}^{\infty} (\sim A_j) \subseteq \sim A_n \\ \Longrightarrow\quad &\mu\left(\sim\bigcup_{j=1}^{\infty} A_j\right) \leqslant \mu(\sim A_n) = 0\,. \end{aligned}$$

Therefore $\sim\left(\bigcup_j A_j\right) \in \mathcal{N}\mu$, and thus again $\bigcup_j A_j \in \mathcal{N}\mu$.

16 Let $\mu = \sum_{q\in\mathbb{Q}} \mu_q$ where μ_q is the Dirac measure at q. So, μ counts the rationals. Now let $\nu = 2\mu$. Then μ and ν are both Borel (since all sets are measurable with respect to counting measure), and they are both infinite on any open set. However, they obviously differ on any finite set of rationals, and thus differ on a closed (and therefore Borel) set.

18 We want to show that if $D \subseteq \mathbb{R}$ and $\mathscr{L}(D) > 0$ then D contains a non-measurable set E. We begin with the well-known Vitali construction of a Lebesgue non-measurable subset M of $[0,1]$. So, for x and y in $\mathbb{R}$, we define an equivalence by $x \equiv y$ if $x - y \in \mathbb{Q}$. Then, we use the Axiom of Choice to form a set $M \subseteq [0,1]$ with exactly one element from each equivalence class of $[0,1]$. To show M is not measurable, consider the sets $M+q$ for the countable q in $\mathbb{Q} \cap [-1,1]$. Then it is easy to see that these sets are disjoint, and that

$$[0,1] \subseteq \bigcup_{q \in \mathbb{Q} \cap [-1,1]} M + q \subseteq [-1,2]\,.$$

But if M were measurable then $M+q$ would be as well, by the translation invariance of $\mathscr{L}$. So, we would have

$$1 = \mathscr{L}([0,1]) \leqslant \sum_{q \in \mathbb{Q} \cap [-1,1]} \mathscr{L}(M+q) = \mathscr{L}\left(\bigcup_{q \in \mathbb{Q} \cap [-1,1]} M + q \right) \leqslant \mathscr{L}([-1,2]) = 3\,.$$

The translation invariance of $\mathscr{L}$ would then imply

$$1 \leqslant \sum_{q \in \mathbb{Q} \cap [-1,1]} \mathscr{L}(M+q) \leqslant 3\,.$$

But since the sum can only be 0 or ∞, this is impossible, and M must be non-measurable.

Now, given $D \subseteq \mathbb{R}$ with $\mathscr{L}(D) > 0$, by countable subadditivity D intersects some interval $[n, n+1]$ in a set of positive measure. So, replacing D by intersection with that interval and translating, we can assume $D \subseteq [0,1]$. Now, with M as constructed above, note that

$$0 < \mathscr{L}(D) = \mathscr{L}\left(\bigcup_{q \in \mathbb{Q} \cap [-1,1]} D \cap (M+q) \right) \leqslant \sum_{q \in \mathbb{Q} \cap [-1,1]} \mathscr{L}\left(D \cap (M+q)\right)\,.$$

Thus, for some $r \in \mathbb{Q} \cap [-1,1]$, $M + r$ intersects D in a set of positive measure. We now set

$$E = D \cap (M+r)\,.$$

We now claim E is non-measurable. Supposing otherwise, then by translation invariance each $E+q$ would be measurable and of positive measure. And, as as for the Vitali set, we could then calculate

$$\infty = \sum_{q \in \mathbb{Q} \cap [-1,1]} \mathscr{L}\left(E+q\right) = \mathscr{L}\left(\bigcup_{q \in \mathbb{Q} \cap [-1,1]} E + q \right) \leqslant \mathscr{L}([-1,2]) = 3\,.$$

This contradiction shows that E is non-measurable, and we're done.